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APPLICATIONS OF FURUTA INEQUALITY BASED ON A SIMPLE CHARACTERIZATION OF CHAOTIC ORDER

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1. Introduction Throughout this note, we consider bounded linear operators acting on a Hilbert space, simply operators. An operator A on H is said to be positive, in symbol $A \geq 0$, if $(Ax, x) \geq 0$ for all $x \in H$. In particular, $A > 0$ means that A is positive and invertible, i.e., $A \geq \alpha$ for some $\alpha > 0$. It is well-known that $A \geq B \geq 0$ does not assure $A^2 \geq B^2$ in general, but the Löwner-Heinz inequality says that the function $t \to t^\alpha$ on $[0, \infty)$ is operator monotone for $0 \leq \alpha \leq 1$, i.e.,

$$A \geq B \geq 0 \quad \text{implies} \quad A^\alpha \geq B^\alpha, \tag{1}$$

cf. [19], [22] and [23]. Furuta [9] gave it an ingenious extension which is called the Furuta inequality, cf. [3,4,5,7,10,11,12,13,14,20,21] and especially [10] for an elementary and one-page proof; [24] for the best possibility of the domaim in which the Furuta inequality holds:

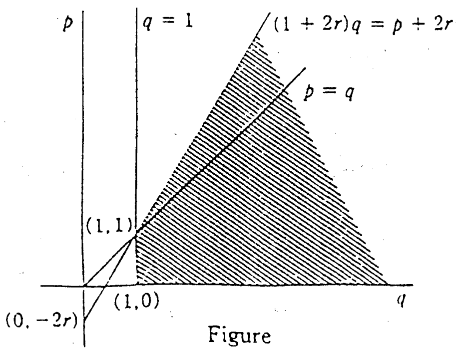
The Furuta inequality. *If $A \geq B \geq 0$, then for each $r \geq 0$*

$$(A^r A^p A^r)^{1/q} \geq (A^r B^p A^r)^{1/q} \tag{2}$$

and

$$(B^r A^p B^r)^{1/q} \geq (B^r B^p B^r)^{1/q} \tag{2'}$$

holds for $p \geq 0$ and $q \geq 1$ with $(1+2r)q \geq p + 2r$.



Figure

Since log t is operator monotone, i.e., log $A \geq$ log B for $A \geq B > 0$, it induces a weaker order $\gg$ among positive invertible operators than the usual one $\geq$, which is called the chaotic order. Now Ando's theorem [1] is rephrased as a characterization of the chaotic order via a form of (2):

Theorem A. *For $A, B > 0$, $A \gg B$ if and only if*

$$(A^{\frac{p}{2}} B^p A^{\frac{p}{2}})^{\frac{1}{2}} \leq A^p \tag{3}$$

holds for all $p \geq 0$.

Afterwards, Theorem A is extended to the following result. In other words, the Furuta inequality is discussed under the chaotic order [5], cf. [7] and [14].

Theorem B. *For $A, B > 0$, $A \gg B$ if and only if*

$$(A^r B^p A^r)^{\frac{2r}{p+2r}} \leq A^{2r} \tag{4}$$

holds for all $p, r \geq 0$.

In this note, we first give a simple characterization of the chaotic order. Precisely, $\log A > \log B$ if and only if there is an $\alpha > 0$ such that

$$A^\alpha > B^\alpha. \tag{5}$$

As an application, we discuss Furuta's type operator inequality implying Theorem B. Next we consider the grand Furuta inequality under the chaotic order. It is given by Furuta [15] as a parameteric formula interpolating the Furuta inequality (2) and the Ando-Hiai one [2; Theorem 3.5] in the following manner : If $A \geq B \geq 0$ and A is invertible, then for each $t \in [0, 1]$,

$$\{A^{\frac{r}{2}}(A^{-\frac{t}{2}}B^pA^{-\frac{t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}} \leq A^{1-t+r} \tag{6}$$

holds for $r \geq t$, $p > 0$ and $s \geq 1$. We here show that if $A, B > 0$ and $\log A > \log B$, then there exists an $\alpha > 0$ such that for each $0 \leq t \leq \alpha$

$$\{A^{\frac{r}{2}}(A^{-\frac{t}{2}}B^pA^{-\frac{t}{2}})^sA^{\frac{r}{2}}\}^{\frac{\alpha-t+r}{(p-t)s+r}} \leq A^{\alpha-t+r} \tag{7}$$

holds for $r \geq t$, $p > 0$ and $s \geq 1$. Finally we also discuss a variant of the grand Furuta inequality under the chaotic order. As a corollary, we have a recent result due to Furuta [17; Cor. 2.5], which is an essential part in the proof of [16; Theorem 2].

2. Characterization of chaotic order. We begin by stating a simple lemma which is the heart of this note:

Lemma 1. *If A and B are selfadjoint and $A > B$, then there exists an $\alpha \in (0, 1]$ such that*

$$e^{\alpha A} > e^{\alpha B}. \tag{8}$$

Proof. The assumption $A > B$ means that $A - B \geq \epsilon > 0$ for some ϵ. We here take $0 < \alpha < \epsilon/(e^{\|A\|} + e^{\|B\|})$ and $\alpha \leq 1$. Then we have

$$\begin{aligned}
e^{\alpha A} - e^{\alpha B} &= \alpha(A - B) + \sum_{n=2}^{\infty} \frac{\alpha^n}{n!}(A^n - B^n) \\
&\geq \alpha\epsilon + \alpha^2 \sum_{n=2}^{\infty} \frac{\alpha^{n-2}}{n!}(A^n - B^n) \\
&\geq \alpha\epsilon - \alpha^2 \| \sum_{n=2}^{\infty} \frac{\alpha^{n-2}}{n!}(A^n - B^n) \| \\
&\geq \alpha\epsilon - \alpha^2 \sum_{n=2}^{\infty} \frac{1}{n!}(\|A\|^n + \|B\|^n) \\
&\geq \alpha(\epsilon - \alpha(e^{\|A\|} + e^{\|B\|})) > 0.
\end{aligned}$$

Lemma 1 implies the following basic inequality :

Corollary 2. *If $A, B > 0$, then* $\log A > \log B$ *if and only if there exists an* $\alpha in(0,1]$ *such that* $A^\alpha > B^\alpha$.

Proof. If $\log A > \log B$, then $A^\alpha > B^\alpha$ for some $\alpha \in (0,1]$ by Lemma 1. Conversely, if $A^\alpha > B^\alpha$ for some $\alpha \in (0,1]$, then $A^\alpha \geq B^\alpha + \delta$ for some $\delta > 0$ and

$$\alpha \log A = \log A^\alpha \geq \log(B^\alpha + \delta) > \log B^\alpha = \alpha \log B.$$

By the above discussion, we have the following simple characterization of the chaotic order:

Theorem 3. *For $A, B > 0$, $A \gg B$, i.e.,* $\log A \geq \log B$, *if and only if for any $\delta \in (0,1]$ there exists an $\alpha = \alpha_\delta > 0$ such that*

$$(e^\delta A)^\alpha > B^\alpha. \tag{9}$$

Proof. Since $A \gg B$ is equivalent to $\log e^\delta A = \log A + \delta > \log B$ for any $\delta > 0$, Corollary 2 implies that $A \gg B$ is equivalent to that for any $\delta > 0$ there exists an $\alpha = \alpha_\delta \in (0,1]$ such that $(e^\delta A)^\alpha > B^\alpha$.

The result in this section is appeared in [6].

3. The Furuta inequality under chaotic order. In this section, we discuss the Furuta inequality under the chaotic order. Such an attempt has been made in our previous papers [5,7] and we obtained Theorem B, which is based on Theorem A. Now we present the following inequality which is more like the Furuta inequality (2) than (4) in Theorem B. Technically speaking, it is just an application of the Furuta inequality (2) as seen below:

Theorem 4. *If $A, B > 0$ and $A \gg B$, then for any $\delta > 0$ there exists an $\alpha = \alpha_\delta \in (0,1]$ such that*

$$(A^r B^p A^r)^{\frac{1}{q}} \leq e^{\frac{\delta p}{q}} A^{\frac{p+2r}{q}} \tag{10}$$

holds for $p \geq 0$, $r \geq 0$ and $q \geq 1$ with $(\alpha + 2r)q \geq p + 2r$.

Proof. By Theorem 3, for any $\delta > 0$ we can choose an $\alpha = \alpha_\delta \in (0,1]$ such that $(e^\delta A)^\alpha > B^\alpha$. Let us put $A_1 = (e^\delta A)^\alpha$, $B_1 = B^\alpha$, $r_1 = \frac{r}{\alpha}$ and $p_1 = \frac{p}{\alpha}$. If p, q and r satisfy the condition stated in Theorem 4, then

$$(1 + 2r_1)q = \frac{1}{\alpha}(\alpha + 2r)q \geq \frac{1}{\alpha}(p + 2r) = p_1 + 2r_1,$$

that is, $\{A_1, B_1; p_1, q, r_1\}$ satisfies all the conditions for implying the Furuta inequality (2). Hence we have

$$\begin{aligned}
(A^r B^p A^r)^{\frac{1}{q}} &= e^{-\frac{2\delta r}{q}} (A_1^{r_1} B_1^{p_1} A_1^{r_1})^{\frac{1}{q}} \\
&\leq e^{-\frac{2\delta r}{q}} A_1^{\frac{p_1+2r_1}{q}} \\
&= e^{-\frac{2\delta r}{q}} (e^\delta A)^{\frac{p+2r}{q}} = e^{\frac{\delta p}{q}} A^{\frac{p+2r}{q}},
\end{aligned}$$

as desired.

Theorem 4 has the following corollary equivalent to Theorem B:

Corollary 5. *If $A, B > 0$ and $A \gg B$, then*

$$(A^r B^p A^r)^{\frac{1}{q}} \le A^{\frac{p+2r}{q}} \tag{4}$$

holds for $p \ge 0$, $r \ge 0$ and $q \ge 1$ with $2rq \ge p + 2r$.

Proof. We first note that if p, q and r satisfy the above condition, then $(\alpha + 2r)q \ge p + 2r$ for all $\alpha > 0$. Hence Theorem 4 implies that for any $\delta > 0$

$$(A^r B^p A^r)^{\frac{1}{q}} \le e^{\frac{\delta p}{q}} A^{\frac{p+2r}{q}}$$

holds for $p \ge 0$, $r \ge 0$ and $q \ge 1$ with $2rq \ge p + 2r$. Taking $\delta \to 0$, we have the required inequality.

Remark. (1) In [5], we proved Theorem B, which is based on Theorem A by applying the Furuta inequality. On the other hand, Corollary 5 (equivalent to Theorem B) follows from Theorem 4, which is an application of the Furuta inequality via Theorem 3. That is, we could give a proof of Theorem B not based on Theorem A, in which Theorem 3 is actually used instead of Theorem A.

(2) We remark that the condition $(\alpha + 2r)q \ge p + 2r$ in Theorem 4 cannot be weakened to $(1 + 2r)q \ge p + 2r$, which is based on an example due to Furuta [8]: As a matter of fact, we put

$$B_1 = \begin{pmatrix} 4 & 5 \\ 5 & 10 \end{pmatrix}^2, \ C_1 = \begin{pmatrix} 1/2 & 0 \\ 0 & 0 \end{pmatrix}^2, A_1 = B_1 + C_1,$$

and $r_1 = 1, p_1 = 3, q_1 = \frac{3}{2}$. Then it is clear that $A_1 \ge B_1 \ge 0$ and

$$(A_1^{r_1} B_1^{p_1} A_1^{r_1})^{\frac{1}{q_1}} \not\le A_1^{\frac{p_1+2r_1}{q_1}}.$$

(Actually $(1 + 2r_1)q_1 \not\ge p_1 + 2r_1$.) Let us put $A = A_1^2, B = B_1^2, r = \frac{1}{2}r_1 = \frac{1}{2}, p = \frac{1}{2}p_1 = \frac{3}{2}$ and $q = q_1 = \frac{3}{2}$. Then $A_1 \ge B_1$ implies $A \gg B$ and $(1 + 2r)q = 3 \ge \frac{5}{2} = p + 2r$. On the other hand, we have

$$(A^r B^p A^r)^{\frac{1}{q}} = (A_1^{r_1} B_1^{p_1} A_1^{r_1})^{\frac{1}{q_1}} \not\le A_1^{\frac{p_1+2r_1}{q_1}} = A^{\frac{p+2r}{q}}.$$

Concluding this section, we present the following characterizations of the chaotic order by summing up the above argument:

Theorem 6. *For $A, B > 0$, the following assertions are mutually equivalent;*

(1) $A \gg B$, i.e., $\log A \gg \log B$.

(2) For any $\delta > 0$ there exists an $\alpha = \alpha_\delta > 0$ such that $(A^r B^p A^r)^{\frac{1}{q}} \le e^{\frac{\delta p}{q}} A^{\frac{p+2r}{q}}$ holds for $p \ge 0$, $r \ge 0$ and $q \ge 1$ with $(\alpha + 2r)q \ge p + 2r$.

(3) $(A^r B^p A^r)^{\frac{1}{q}} \le A^{\frac{p+2r}{q}}$ *holds for* $p \ge 0$, $r \ge 0$ *and* $q \ge 1$ *with* $2rq \ge p + 2r$.

(4) $(A^r B^p A^r)^{\frac{2r}{p+2r}} \le A^{2r}$ *holds for all* $p, r \ge 0$.

(5) $(A^{\frac{p}{2}} B^p A^{\frac{p}{2}})^{\frac{1}{2}} \le A^p$ *holds for all* $p \ge 0$.

4. The grand Furuta inequality under chaotic order. Now the grand Furuta inequality [15] is mentioned that if $A \ge B \ge 0$ and $A > 0$, then for $0 \le t \le 1$

$$\{A^{\frac{r}{2}}(A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}})^s A^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}} \le A^{1-t+r} \tag{11}$$

holds for $r \ge t$, $p \ge 1$ and $s \ge 1$. Taking $t = 1$ and $s = r$, we have the Ando-Hiai inequality [2; Theorem 3.5] as a special case. That is, the following inequality holds under the same assumption as above;

$$A^r \ge \{A^{r/2}(A^{-1/2} B^p A^{-1/2})^r A^{r/2}\}^{1/p} \tag{12}$$

for $p, r \ge 1$. On the other hand, taking $t = 0$ and $s = 1$, we have the Furuta inequality (2). Namely the grand Furuta inequality is understood as a parametric formula interpolating the Ando-Hiai inequality and the inequality by himself.

Now we discuss the grand Furuta inequality under the chaotic order. For simplicity, we assume that $\log A > \log B$. The following theorem is an application of the grand Furuta inequality, too.

Theorem 7. *If* $A, B > 0$ *and* $\log A > \log B$, *then there exists an* $\alpha > 0$ *such that for* $0 \le t \le \alpha$

$$\{A^{\frac{r}{2}}(A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}})^s A^{\frac{r}{2}}\}^{\frac{\alpha-t+r}{(p-t)s+r}} \le A^{\alpha-t+r} \tag{13}$$

holds for $r \ge t$, $p \ge \alpha$ *and* $s \ge 1$.

Proof. By Corollary 2, we can suppose that $A^\alpha > B^\alpha$ for some $0 < \alpha \le 1$. For $0 \le t \le \alpha \le p$, we put $A_1 = A^\alpha$, $B_1 = B^\alpha$,; $t_1 = \frac{t}{\alpha}$, $r_1 = \frac{r}{\alpha}$, $p_1 = \frac{p}{\alpha}$. Since $0 \le t \le \alpha \le p$, $r \ge t$ and $s \ge 1$, we have $0 \le t_1 \le 1$, $r_1 \ge t_1$, $p_1 \ge 1$ (and $s \ge 1$). Therefore the grand Furuta inequality assures that

$$\begin{aligned}
&\{A^{\frac{r}{2}}(A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}})^s A^{\frac{r}{2}}\}^{\frac{\alpha-t+r}{(p-t)s+r}} \\
&= \{A_1^{\frac{r_1}{2}}(A_1^{-\frac{t_1}{2}} B_1^{p_1} A_1^{-\frac{t_1}{2}})^s A_1^{\frac{r_1}{2}}\}^{\frac{1-t_1+r_1}{(p_1-t_1)s+r_1}} \\
&\le A_1^{1-t_1+r_1} \\
&= A^{\alpha-t+r}.
\end{aligned}$$

So the proof is complete.

Incidentally, (13) implies the following inequality on the chaotic order:

$$A^{\frac{r}{2}}(A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}})^s A^{\frac{r}{2}} \ll A^{(p-t)s+r} \tag{14}$$

for $r \ge t$, $p > 0$ and $s \ge 1$.

Remark. Exactly the grand Furuta inequality is expressed as the monotonicity of an operator function as follows:

The grand Furuta inequality. *If $A \geq B \geq 0$ and A is invertible, then for each $t \in [0,1]$,*

$$F_{p,t}(A,B,r,s) = A^{-r/2}\{A^{r/2}(A^{-t/2}B^pA^{-t/2})^sA^{r/2}\}^{\frac{1-t+r}{(p-t)s+r}}A^{-r/2}$$

is a decreasing function of both r and s for all $r \geq t$, $p \geq 1$ and $s \geq 1$.

Therefore (11) is a special case of the grand Furuta inequality, more precisely. We now remark that a chaotic version of the grand Furuta inequality can be considered, which will be done in the forthcoming paper by T.Furuta [18].

5. A variant of the grand Furuta inequality. Very recently, Furuta proposed in [16; Cor. 2.5] the following variant of the grand Furuta inequality (11) related to a parallelism among Furuta's type operator inequalities. Actually it is the opposite of (11) on the sign of t:

Theorem C. *If $A \geq B \geq 0$, then*

$$\{A^{\frac{r}{2}}(A^{\frac{t}{2}}B^pA^{\frac{t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1+t+r}{(p+t)s+r}} \leq A^{1+t+r} \tag{15}$$

holds for $t \geq 0$, $r \geq 0$, $p \geq 1$ and $s \geq 1$.

By a similar way to the preceding section, a chaotic version of Theorem C is given and also an application of Theorem 3:

Theorem 8. *If $A, B > 0$ and $A \gg B$, then for any $\delta > 0$ there exists an $\alpha > 0$ such that*

$$\{A^{\frac{r}{2}}(A^{\frac{t}{2}}B^pA^{\frac{t}{2}})^sA^{\frac{r}{2}}\}^{\frac{\alpha+t+r}{(p+t)s+r}} \leq (e^{\frac{\delta ps}{(p+t)s+r}}A)^{\alpha+t+r}. \tag{16}$$

holds for $t \geq 0$, $r \geq 0$, $p > 0$ and $s \geq 1$.

Proof. First of all, for a given $\delta > 0$ there exists an $\alpha = \alpha_\delta > 0$ such that

$$A_1 = (e^\delta A)^\alpha > B^\alpha = B_1$$

by Theorem 3, and we can add to the condition $0 < \alpha \leq p$ by the Löwner-Heinz inequality. Putting $t_1 = \frac{t}{\alpha}$, $r_1 = \frac{r}{\alpha}$ and $p_1 = \frac{p}{\alpha}$, Theorem C implies that

$$\begin{aligned}
&\{A^{\frac{r}{2}}(A^{\frac{t}{2}}B^pA^{\frac{t}{2}})^sA^{\frac{r}{2}}\}^{\frac{\alpha+t+r}{(p+t)s+r}} \\
&= e^{-\frac{\delta(r+ts)(\alpha+t+r)}{(p+t)s+r}}\{A_1^{\frac{r_1}{2}}(A_1^{\frac{t_1}{2}}B_1^{p_1}A_1^{\frac{t_1}{2}})^sA_1^{\frac{r_1}{2}}\}^{\frac{1+t_1+r_1}{(p_1+t_1)s+r_1}} \\
&\leq e^{-\frac{\delta(r+ts)(\alpha+t+r)}{(p+t)s+r}}A_1^{1+t_1+r_1} \\
&= (e^{\frac{\delta ps}{(p+t)s+r}}A)^{\alpha+t+r}.
\end{aligned}$$

So the proof is complete.

Corollary 9. (Furuta) [17] *If $A, B > 0$ and $A \gg B$, then*

$$(17) \qquad \{A^{\frac{2\beta+(u\gamma+p)s}{2}}(A^{\frac{u\gamma}{2}}B^pA^{\frac{u\gamma}{2}})^sA^{\frac{2\beta+(u\gamma+p)s}{2}}\}^{\frac{1}{2}} \le A^{(u\gamma+p)s+\beta}$$

holds for $p \ge u > 0$, $s \ge 1$, $1 \ge \gamma \ge 0$ and $\beta \ge -u\gamma$.

Proof. If we put $t = u\gamma$ and $r = 2\beta + (u\gamma + p)s$, then $t \ge 0$ and

$$r \ge 2\beta + u\gamma + p = 2(\beta + u\gamma) + (p - u\gamma) \ge 0.$$

Hence it follows from Theorem 8 that for any $\delta > 0$ there exists an $\alpha > 0$ such that

$$(18) \qquad \{A^{\frac{r}{2}}(A^{\frac{t}{2}}B^pA^{\frac{t}{2}})^sA^{\frac{r}{2}}\}^{\frac{\alpha+t+r}{(p+t)s+r}} \le (e^{\frac{\delta ps}{(p+t)s+r}}A)^{\alpha+t+r}$$

holds for $t \ge 0$, $r \ge 0$, $p > 0$ and $s \ge 1$.

Since $\alpha > 0$ and $u\gamma + \beta \ge 0$, we have

$$\frac{(p+t)s+r}{2(\alpha+t+r)} \le \frac{(p+u\gamma)s + 2\beta + (u\gamma+p)s}{2(u\gamma + 2\beta + (u\gamma+p)s)} = \frac{\beta + (u\gamma+p)s}{(u\gamma+\beta) + (\beta + (u\gamma+p)s)} \le 1.$$

Taking the power $\frac{(p+t)s+r}{2(\alpha+t+r)}$ on both sides of (18), it follows from (1) that

$$\begin{aligned}
&\{A^{\frac{2\beta+(u\gamma+p)s}{2}}(A^{\frac{u\gamma}{2}}B^pA^{\frac{u\gamma}{2}})^sA^{\frac{2\beta+(u\gamma+p)s}{2}}\}^{\frac{1}{2}} \\
&= \{A^{\frac{2\beta+(u\gamma+p)s}{2}}(A^{\frac{u\gamma}{2}}B^pA^{\frac{u\gamma}{2}})^sA^{\frac{2\beta+(u\gamma+p)s}{2}}\}^{\frac{\alpha+t+r}{(p+t)s+r}\cdot\frac{(p+t)s+r}{2(\alpha+t+r)}} \\
&\le \{e^{\frac{\delta ps}{(p+t)s+r}}A\}^{(\alpha+t+r)\cdot\frac{(p+t)s+r}{2(\alpha+t+r)}} \\
&= e^{\frac{\delta ps}{2}}A^{(p+u\gamma)s+\beta}.
\end{aligned}$$

Hence we have the conclusion by taking $\delta \to 0$.

Remark. Furuta proved in [15; Theorem 2] that if $A \gg B$, then there exists a partial isometry U satisfying

$$(19) \qquad A^{\frac{\beta}{2}}(A^{\frac{u\gamma}{2}}B^pA^{\frac{u\gamma}{2}})^sA^{\frac{\beta}{2}} \le U^*A^{(u\gamma+p)s+\beta}U$$

holds for $p \ge u > 0$, $s \ge 1$, $1 \ge \gamma \ge 0$ and $\beta \ge -u\gamma$. Consequently he obtained a generalized Kosaki trace inequality; if $A \gg B$, then for a continuous increasing function f on $\mathbb{R}_+$ with $f(0) = 0$

$$Tr(f(A^{\frac{\beta}{2}}(A^{\frac{u\gamma}{2}}B^pA^{\frac{u\gamma}{2}})^sA^{\frac{\beta}{2}})) \le Tr(f(A^{(u\gamma+p)s+\beta}))$$

holds for $p \ge u > 0$, $s \ge 1$, $1 \ge \gamma \ge 0$ and $\beta \ge -u\gamma$.

Finally we make a path from Corollary 9 to (19) clear. As in [16], we use the lemma that there is a partial isometry U such that $XSX \le U^*SU$ for given $S \ge 0$ and $0 \le X \le 1$. Since Corollary 9 says that

$$X = A^{-\frac{(u\gamma+p)s+\beta}{2}}\{A^{\frac{2\beta+(u\gamma+p)s}{2}}(A^{\frac{u\gamma}{2}}B^pA^{\frac{u\gamma}{2}})^sA^{\frac{2\beta+(u\gamma+p)s}{2}}\}^{\frac{1}{2}}A^{-\frac{(u\gamma+p)s+\beta}{2}}$$

is a contraction, i.e., (17), if we choose $S = A^{(u\gamma+p)s+\beta}$, then we can apply it to X and S. Namely we have (19);

$$A^{\frac{\beta}{2}}(A^{\frac{u\gamma}{2}}B^pA^{\frac{u\gamma}{2}})^sA^{\frac{\beta}{2}} = XSX \le U^*SU = U^*A^{(u\gamma+p)s+\beta}U$$

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